Off-axis electric field of a ring of charge

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1 Setting

A generic point \( \mathbf{p}_0 \) on a ring laying on the XY plane can be described as

\[
\mathbf{p}_0 = R \cos \alpha \hat{i} + R \sin \alpha \hat{j} \quad \alpha = [0, 2\pi)
\]

where \( R \) is the radius of the ring, \( \hat{i} \) and \( \hat{j} \) are unit vectors and \( \alpha \) is the parametric angle.

We want to find the off-axis electric field strength in point \( \mathbf{p} \). As the ring is symmetric we can place the point on the XZ plane and describe it through radial distance \( r \) from the axis of the ring and axial distance \( a \) along the axis of the ring. With the angle \( \theta \) between normal vector \( \hat{k} \) and point at \( \mathbf{p} \) we can describe the following relations:

\[
\begin{align*}
\mathbf{p} &= (r, 0, a) = r \hat{i} + a \hat{k} \\
\cos \theta &= \frac{\mathbf{p} \cdot \hat{k}}{||\mathbf{p}||} \\
r &= ||\mathbf{p}|| \sin \theta \\
a &= ||\mathbf{p}|| \cos \theta \\
\mathbf{p} - \mathbf{p}' &= (r - R \cos \alpha) \hat{i} - R \sin \alpha \hat{j} + a \hat{k} \\
||\mathbf{p} - \mathbf{p}'|| &= \sqrt{(r - R \cos \alpha)^2 + R^2 \sin^2 \alpha + a^2} \\
&= \sqrt{r^2 - 2rR \cos \alpha + R^2 \cos^2 \alpha + R^2 \sin^2 \alpha + a^2} \\
&= \sqrt{r^2 + a^2 - 2rR \cos \alpha}
\end{align*}
\]

2 Electric potential at point \( \mathbf{p} \)

Electric potential of a point of charge is

\[
\varphi = \frac{q}{4 \pi \varepsilon_0 r}
\]

Let \( Q \) be the total charge on the ring and let the charge be uniformly distributed. Integrating over the ring of charge gives us

\[
\varphi = \frac{1}{4 \pi \varepsilon_0} \frac{Q}{2 \pi R} \int_0^{2\pi R} ds \frac{1}{||\mathbf{p} - \mathbf{p}'||}
\]
\[ \frac{\partial K(k)}{\partial k} = \frac{E(k) - (1 - k^2)K(k)}{k(1 - k^2)} \]

Deriving the axial component of the electric field

\[ \frac{\partial q}{\partial a} = 2a \]
\[ \frac{\partial k}{\partial a} = \frac{1}{2} \frac{4rR}{q^2} \]
\[ = \frac{ak}{q} \]
\[
\frac{\partial K(k)}{\partial a} = \frac{\partial K(k)}{\partial k} \frac{\partial k}{\partial a}
\]

\[
\frac{\partial q}{\partial r} = 2(r + R)
\]

\[
\frac{\partial k}{\partial r} = \frac{1}{2} \frac{4Rq - (4rR)2(r + R)}{\sqrt{q}}
\]

\[
\frac{dK(k)}{dk} = \frac{E(k) - \frac{(1-k^2)K(k)}{k(1-k^2)}}{k(1-k^2)}
\]

\[
\frac{dK(k)}{dr} = \frac{dK(k)}{dk} \frac{dk}{dr}
\]

\[
\frac{\partial \phi}{\partial r} = \frac{Q}{4\pi \varepsilon_0 \pi} \left[ \frac{E(k) - \frac{(1-k^2)K(k)}{k(1-k^2)}}{k(1-k^2)} \left( \frac{2R - k^2(r + R)}{kq} \right) \sqrt{q} - \frac{1}{2} \sqrt{q} \frac{2(r + R)K(k)}{q} \right]
\]

\[
= \frac{Q}{4\pi \varepsilon_0 \pi} \left[ \frac{(E(k) - (1-k^2)K(k))(2R - k^2(r + R))}{k^2(1-k^2)\sqrt{q}} \right]
\]

\[
= \frac{Q}{4\pi \varepsilon_0 \pi} \left[ \frac{(E(k) - (1-k^2)K(k))(2R - k^2(r + R))}{k^2(1-k^2)\sqrt{q}} - \frac{(r + R)K(k)}{\sqrt{q}} \right]
\]

The same for the radial component:
\[
\begin{align*}
E_r &= \frac{Q}{4\pi\varepsilon_0 \pi k^2 (1 - k^2) q^2} \left[ 2RE(k) - E(k) k^2 (r + R) - 2R(1 - k^2) K(k) \right] \\
E_a &= \frac{Q}{4\pi\varepsilon_0 \pi k^2 (1 - k^2) q^2} \left[ 2RE(k) - k^2 rE(k) - k^2 RE(k) - 2RK(k) + 2R^2 K(k) \right]
\end{align*}
\]

4 Results

Here are the electric field radial and axial components for the off-axis electric field of a ring of charge:

\[
E_r = \frac{Q}{4\pi\varepsilon_0 \pi q^2 (1 - \mu) \mu} \left( 2RK\sqrt{\mu}(1 - \mu) - E(\sqrt{\mu})(2R - \mu(r + R)) \right)
\]

\[
E_a = \frac{Q}{4\pi\varepsilon_0 \pi q^2 (1 - \mu)} aE(\sqrt{\mu}) \]

where

\[
q = r^2 + R^2 + a^2 + 2rR
\]

\[
\mu = \frac{4rR}{q}
\]

and \( K(\sqrt{\mu}) \) is the complete elliptic integral of the first kind and \( E(\sqrt{\mu}) \) is the complete elliptic integral of the second kind.

As a bonus here are the complementary equations for the off-axis magnetic field of a ring of current:

\[
B_r = \frac{\mu_0 I}{4\pi \sqrt{q}} \left( \frac{2a}{\sqrt{q} r} \left[ E(\sqrt{q}) \frac{q - 2rR}{q - 4rR} - K(\sqrt{q}) \right] \right)
\]

\[
B_a = \frac{\mu_0 I}{4\pi \sqrt{q}} \left[ E(\sqrt{q}) \frac{R^2 - r^2 - a^2}{q - 4rR} + K(\sqrt{q}) \right]
\]

5 Sources

The following material was used in compiling this paper:

- Wikipedia
  http://en.wikipedia.org/wiki/Main_Page
- “Off-axis electric field of a ring of charge”
  Fredy R. Zypman,
• “Elliptic Integrals, Elliptic Functions and Theta Functions”
  Prof. J. R. Culham
  http://www.mhtlab.uwaterloo.ca/courses/me755/

• Off-axis Field Due to a Current Loop
  Eric Dennison
  http://www.netdenizen.com/emagnet/offaxis/iloopoffaxis.htm