Off-axis electric field of a ring of charge

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1 Setting

A generic point $\vec{p'}$ on a ring laying on the XY plane can be described as

$$\vec{p'} = R\cos\alpha\hat{i} + R\sin\alpha\hat{j} \quad \alpha = [0, 2\pi)$$

where R is the radius of the ring, $\hat{i}$ and $\hat{j}$ are unit vectors and α is the parametric angle. We want to find the off-axis electric field strength in point $\vec{p}$. As the ring is symmetric we can place the point on the XZ plane and describe it through radial distance r from the axis of the ring and axial distance a along the axis of the ring. With the angle θ between normal vector $\hat{k}$ and point at $\vec{p}$ we can describe the following relations:

$$\begin{aligned}
\vec{p} &= (r, 0, a) = r\hat{i} + a\hat{k} \\
\cos\theta &= \frac{\vec{p}\cdot\hat{k}}{|\vec{p}|} \\
r &= |\vec{p}|\sin\theta \\
a &= |\vec{p}|\cos\theta \\
\vec{p} - \vec{p'} &= (r - R\cos\alpha)\hat{i} - R\sin\alpha\hat{j} + a\hat{k} \\
|\vec{p} - \vec{p'}| &= \sqrt{(r - R\cos\alpha)^2 + R^2\sin^2\alpha + a^2} \\
&= \sqrt{r^2 - 2rRcos\alpha + R^2cos^2\alpha + R^2sin^2\alpha + a^2} \\
&= \sqrt{r^2 + R^2 + a^2 - 2rR\cos\alpha}
\end{aligned}$$

2 Electric potential at point $\vec{p}$

Electric potential of a point of charge is

$$\varphi = \frac{q}{4\pi\varepsilon_0 r}$$

Let Q be the total charge on the ring and let the charge be uniformly distributed. Integrating over the ring of charge gives us

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2\pi R}\int_0^{2\pi R}\frac{ds}{|\vec{p} - \vec{p'}|}$$

$$
\begin{aligned}
&= \frac{Q}{4\pi\varepsilon_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{\sqrt{r^2+R^2+a^2-2rR\cos\alpha}} \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{1}{\pi}\int_0^{\pi}\frac{d\alpha}{\sqrt{r^2+R^2+a^2-2rR\cos\alpha}} \\
&\qquad (\beta = \frac{\pi-\alpha}{2} \Rightarrow \alpha = \pi - 2\beta,\; d\alpha = -2d\beta) \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\beta}{\sqrt{r^2+R^2+a^2+2rR(1-2\sin^2\beta)}} \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\frac{1}{\sqrt{q}}\int_0^{\frac{\pi}{2}}\frac{d\beta}{\sqrt{1-\frac{4rR}{q}\sin^2\beta}} \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\frac{1}{\sqrt{q}}\int_0^{\frac{\pi}{2}}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\frac{K(k)}{\sqrt{q}} \\
&\qquad \text{where} \\
q &= r^2+R^2+a^2+2rR \\
k &= \sqrt{\frac{4rR}{q}}
\end{aligned}
$$

and $K(k)$ is the complete elliptic integral of the first kind.

3 Electric field

Electric field is the negative gradient of the electric potential:

$$
\begin{aligned}
\varphi &= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\frac{K(k)}{\sqrt{q}} \\
\vec{E} &= -\nabla\varphi = \left(-\frac{\partial\varphi}{\partial r}, 0, -\frac{\partial\varphi}{\partial a}\right)
\end{aligned}
$$

This requires derivation of the elliptic integral function:

$$
\frac{\partial K(k)}{\partial k} = \frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}
$$

Deriving the axial component of the electric field

$$
\begin{aligned}
\frac{\partial q}{\partial a} &= 2a \\
\frac{\partial k}{\partial a} &= \frac{1}{2}\frac{1}{\sqrt{\frac{4rR}{q}}}\frac{-(4rR)(2a)}{q^2} \\
&= -\frac{ak}{q}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial K(k)}{\partial a} &= \frac{\partial K(k)}{\partial k}\frac{\partial k}{\partial a} \\
\frac{\partial \varphi}{\partial a} &= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\left(-\frac{ak}{q}\right)\sqrt{q}-\frac{1}{2}\frac{1}{\sqrt{q}}2aK(k)}{q}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{-\frac{E(k)-(1-k^2)K(k)}{(1-k^2)}\frac{a}{\sqrt{q}}-\frac{aK(k)}{\sqrt{q}}}{q}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2a}{\pi}\left[\frac{-\frac{E(k)-(1-k^2)K(k)+K(k)(1-k^2)}{(1-k^2)\sqrt{q}}}{q}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2a}{\pi}\left[-\frac{E(k)}{(1-k^2)q^{\frac{3}{2}}}\right] \\
&= -\frac{Q}{4\pi\varepsilon_0}\frac{2a}{\pi}\frac{E(k)}{q^{\frac{3}{2}}(1-k^2)}
\end{aligned}
$$

The same for the radial component:

$$
\begin{aligned}
\frac{\partial q}{\partial r} &= 2(r+R) \\
\frac{\partial k}{\partial r} &= \frac{1}{2}\frac{1}{\sqrt{\frac{4rR}{q}}}\frac{4Rq-(4rR)2(r+R)}{q^2} \\
&= \frac{1}{\sqrt{\frac{4rR}{q}}}\frac{2R-k^2(r+R)}{q} \\
&= \frac{2R-k^2(r+R)}{kq} \\
\frac{dK(k)}{dk} &= \frac{E(k)-(1-k^2)K(k)}{k(1-k^2)} \\
\frac{\partial K(k)}{\partial r} &= \frac{dK(k)}{dk}\frac{\partial k}{\partial r} \\
\frac{\partial \varphi}{\partial r} &= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\left(\frac{2R-k^2(r+R)}{kq}\right)\sqrt{q}-\frac{1}{2}\frac{1}{\sqrt{q}}2(r+R)K(k)}{q}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{\frac{(E(k)-(1-k^2)K(k))(2R-k^2(r+R))}{k^2(1-k^2)\sqrt{q}}-\frac{(r+R)K(k)}{\sqrt{q}}}{q}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{\frac{(E(k)-(1-k^2)K(k))(2R-k^2(r+R))-(r+R)K(k)k^2(1-k^2)}{k^2(1-k^2)\sqrt{q}}}{q}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{(E(k)-(1-k^2)K(k))(2R-k^2(r+R))-(1-k^2)K(k)k^2(r+R)}{k^2(1-k^2)q^{\frac{3}{2}}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi}\left[\frac{2RE(k)-E(k)k^2(r+R)-2R(1-k^2)K(k)}{k^2(1-k^2)q^{\frac{3}{2}}}\right] \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi k^2(1-k^2)q^{\frac{3}{2}}}(2RE(k)-k^2(r+R)E(k)-2R(1-k^2)K(k)) \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi k^2(1-k^2)q^{\frac{3}{2}}}(2RE(k)-k^2rE(k)-k^2RE(k)-2RK(k)+2Rk^2K(k)) \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi k^2(1-k^2)q^{\frac{3}{2}}}(E(k)(2R-k^2(r+R))-2RK(k)(1-k^2))
\end{aligned}
$$

4 Results

Here are the electric field radial and axial components for the off-axis electric field of a ring of charge:

$$
\begin{aligned}
E_r &= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi q^{\frac{3}{2}}(1-\mu)}\frac{1}{\mu}(2RK(\sqrt{\mu})(1-\mu)-E(\sqrt{\mu})(2R-\mu(r+R))) \\
E_a &= \frac{Q}{4\pi\varepsilon_0}\frac{2}{\pi q^{\frac{3}{2}}(1-\mu)}aE(\sqrt{\mu})
\end{aligned}
$$

where

$$
\begin{aligned}
q &= r^2+R^2+a^2+2rR \\
\mu &= \frac{4rR}{q}
\end{aligned}
$$

and $K(\sqrt{\mu})$ is the complete elliptic integral of the first kind and $E(\sqrt{\mu})$ is the complete elliptic integral of the second kind.

As a bonus here are the complementary equations for the off-axis magnetic field of a ring of current:

$$
\begin{aligned}
B_r &= \frac{\mu_0}{4\pi}I\frac{2}{\sqrt{q}}\frac{a}{r}\left[E(\sqrt{\mu})\frac{q-2rR}{q-4rR}-K(\sqrt{\mu})\right] \\
B_a &= \frac{\mu_0}{4\pi}I\frac{2}{\sqrt{q}}\left[E(\sqrt{\mu})\frac{R^2-r^2-a^2}{q-4rR}+K(\sqrt{\mu})\right]
\end{aligned}
$$

5 Sources

The following material was used in compiling this paper:

- Wikipedia
  http://en.wikipedia.org/wiki/Main_Page
- "Off-axis electric field of a ring of charge"
  Fredy R. Zypman,
  http://www.physics.buffalo.edu/~sen/documents/field_by_charged_ring.pdf

- “Elliptic Integrals, Elliptic Functions and Theta Functions”
  Prof. J. R. Culham
  http://www.mhtlab.uwaterloo.ca/courses/me755/
- Off-axis Field Due to a Current Loop
  Eric Dennison
  http://www.netdenizen.com/emagnet/offaxis/iloopoffaxis.htm