Electric potential and electric field of a line segment of charge with variable charge distribution

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1 Line segment and charge density

Lets define the line segment through starting and end points $\vec{p}_0$ and $\vec{p}_1$. We'll also define the line vector $\vec{l}$ and normal $\vec{n}$:

$$
\begin{aligned}
\vec{p}_0 &= x_0\hat{i} + y_0\hat{j} + z_0\hat{k} \\
\vec{p}_1 &= x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \\
\vec{l} &= \vec{p}_1 - \vec{p}_0 \\
\hat{n} &= \frac{\vec{l}}{|\vec{l}|} \\
&= x_n\hat{i} + y_n\hat{j} + z_n\hat{k}
\end{aligned}
$$

Lets define a point of interest $\vec{p}$ and the closest point $\vec{p}_l$ on the line:

$$
\begin{aligned}
\vec{p} &= x\hat{i} + y\hat{j} + z\hat{k} \\
\vec{p}_l &= \vec{p}_0 + (\hat{n} \cdot (\vec{p} - \vec{p}_0))\hat{n} \\
\vec{r}_0 &= \vec{p} - \vec{p}_0 \\
\vec{r}_1 &= \vec{p} - \vec{p}_1
\end{aligned}
$$

Lets define the distance between the point and the line as d, and distances a and b from the line segment endpoints:

$$
\begin{aligned}
d &= |\vec{p} - \vec{p}_l| \\
a &= -\hat{n} \cdot (\vec{p} - \vec{p}_0) \\
&= -x_n(x - x_0) - y_n(y - y_0) - z_n(z - z_0) \\
b &= -\hat{n} \cdot (\vec{p} - \vec{p}_1) \\
&= -x_n(x - x_1) - y_n(y - y_1) - z_n(z - z_1)
\end{aligned}
$$

Let the charge densities at the start of the line be ρ_0 and at the end of the line ρ_1 with linear transition between the two points. Lets define parametric functions $r(t)$ and $\rho(t)$ corresponding to the distance of the point from the line and the charge density at the given point:

$$\begin{aligned}
r(t) &= \sqrt{d^2+t^2} \\
\rho(t) &= \rho_0 + \frac{\rho_1-\rho_0}{|\vec{l}|}(t-a) \\
&= \rho_0 - a\frac{\rho_1-\rho_0}{|\vec{l}|} + \frac{\rho_1-\rho_0}{|\vec{l}|}t \\
&= \rho_0 - av + vt \\
&= u + vt \\
t &= [a,b)
\end{aligned}$$

2 The electric potential

The electric potential of a point charge q, at a distance r from the charge is defined as

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

To get the electric potential of a line of charge at point $\vec{p}$ we have to integrate over the line collecting individual charge elements ρdt:

$$\begin{aligned}
\varphi &= \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{u+vt}{\sqrt{d^2+t^2}}dt \\
&= \frac{1}{4\pi\varepsilon_0}\left(u\ln(t+\sqrt{d^2+t^2}) + v\sqrt{d^2+t^2}\right)\Big|_a^b \\
&= \frac{1}{4\pi\varepsilon_0}\left(u\ln\frac{b+\sqrt{d^2+b^2}}{a+\sqrt{d^2+a^2}} + v\left(\sqrt{d^2+b^2}-\sqrt{d^2+a^2}\right)\right) \\
&= \frac{1}{4\pi\varepsilon_0}\left(u\ln\frac{b+|\vec{r}_1|}{a+|\vec{r}_0|} + v\left(|\vec{r}_1|-|\vec{r}_0|\right)\right)
\end{aligned}$$

There is one problem with this result. In case d is 0 and a is negative the equation has no valid value. The solution is to integrate in reverse direction. I must admit this hack shows I'm a bit math-challenged.

3 The electric field

Electric field is defined as the negative gradient of the electric potential

$$
\begin{aligned}
\vec{E} &= -\nabla\varphi \\
&= -\frac{\partial\varphi}{\partial x}\hat{\imath} - \frac{\partial\varphi}{\partial y}\hat{\jmath} - \frac{\partial\varphi}{\partial z}\hat{k}
\end{aligned}
$$

Derivatives of the various components:

$$
\begin{aligned}
\frac{\partial a}{\partial x} &= -x_n \\
\frac{\partial b}{\partial x} &= -x_n \\
\frac{\partial u}{\partial x} &= x_n v \\
\frac{\partial|\vec{r}_0|}{\partial x} &= \frac{x - x_0}{|\vec{r}_0|} = \Delta x_0 \\
\frac{\partial|\vec{r}_1|}{\partial x} &= \frac{x - x_1}{|\vec{r}_1|} = \Delta x_1
\end{aligned}
$$

And the potential

$$
\begin{aligned}
\frac{\partial\varphi}{\partial x} &= \frac{1}{4\pi\varepsilon_0}\left(x_n v \ln\frac{b+|\vec{r}_1|}{a+|\vec{r}_0|} + v(\Delta x_1 - \Delta x_0) + \right. \\
&\left. + u\left(\frac{\Delta x_1 - x_n}{b+|\vec{r}_1|} - \frac{\Delta x_0 - x_n}{a+|\vec{r}_0|}\right)\right)
\end{aligned}
$$

The electric field follows from here:

$$
\begin{aligned}
\vec{E} &= -\frac{1}{4\pi\varepsilon_0}\left(v\left(\ln\frac{b+|\vec{r}_1|}{a+|\vec{r}_0|}\hat{n} + \hat{r}_1 - \hat{r}_0\right) + \right. \\
&\left. + u\left(\frac{\hat{r}_1 - \hat{n}}{b+|\vec{r}_1|} - \frac{\hat{r}_0 - \hat{n}}{a+|\vec{r}_0|}\right)\right)
\end{aligned}
$$

4 Testing

In figures 1 and 2 you can see the electric potential and electric field lines of a line of charge where on one side the charge distribution is 0. In figure 3 you can see the e-field lines of a line of charge with polar charge distribution - that is on one side of the line the charge is negative and on the other side it's positive.

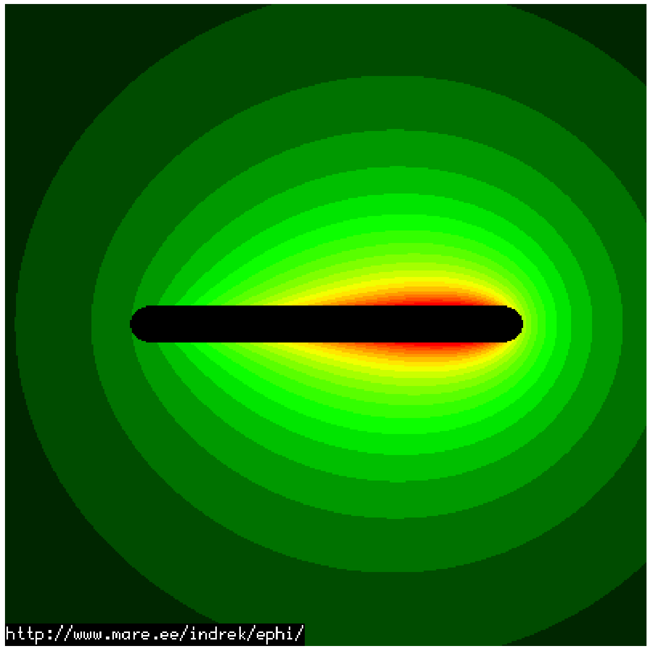

Figure 1: Electric potential of an asymmetric line of charge

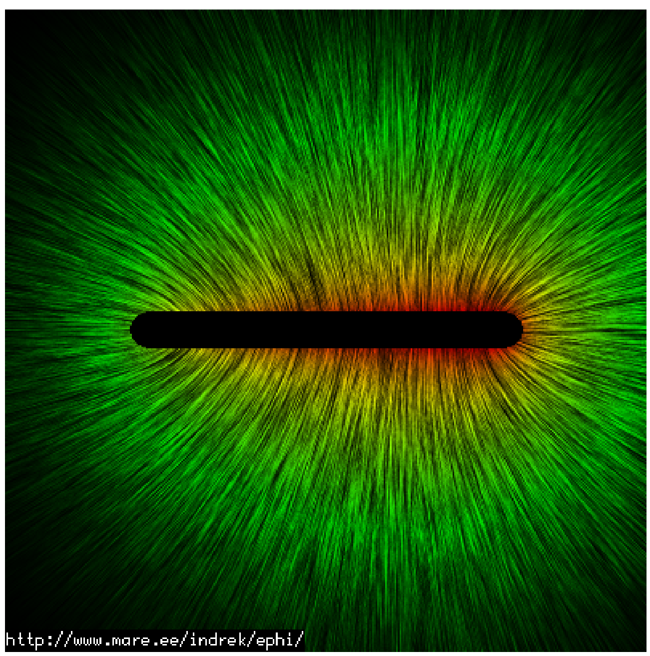

Figure 2: E-field lines of an asymmetric line of charge

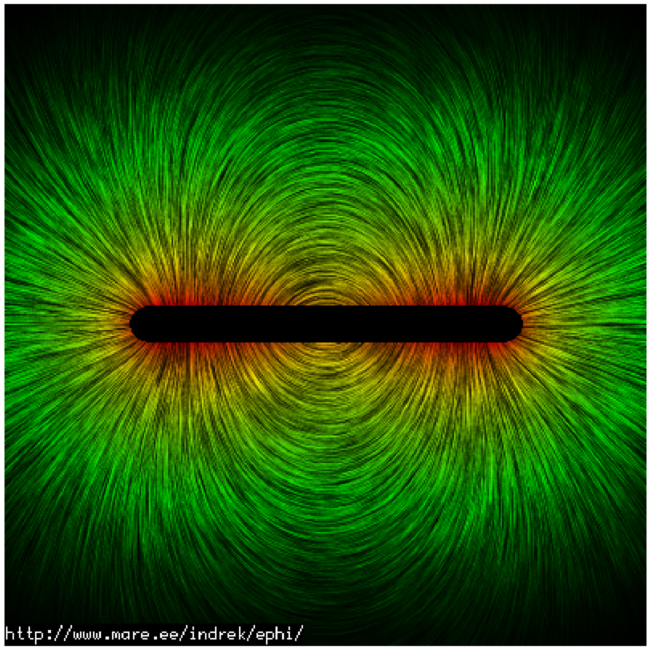

Figure 3: E-field lines of a polar charge distribution

5 Disclaimer

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