Coulomb law, electric potential and Biot-Savart law differentiations for tricubic interpolation

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1 Fields and potential of a charge

The electric field of a stationary point charge is defined as

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

The magnetic field approximation of a charge moving at constant non-relativistic speed (meaning $v^2 \ll c^2$) is defined [1] as

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q}{r^2} \mathbf{v} \times \hat{\mathbf{r}}$$

This can be defined through the electric field

$$\begin{aligned} \mathbf{B} &= \mu_0 \varepsilon_0 \mathbf{v} \times \mathbf{E} \\ &= \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \end{aligned}$$

The electric potential of a stationary charge stands as

$$\phi_E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

When calculating the field of many particles in a system it often makes sense to precalculate the field values into a grid and interpolate from that grid for further applications. One such interpolation method is tricubic interpolation. Tricubic interpolation requires the calculation of first, second and third degree partial derivatives of the values being interpolated. Numeric differentiation can be used but in case of the Coulomb or Biot-Savart law it might make sense to use analytic forms as they are both more precise and faster to calculate. The tricubic interpolation method interpolates each of the vector components separately, so the derivatives are required for all vector components.

2 Differentiation

Differentiation of the magnetic field comes down to the electric field, for example:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial x} &= \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial x}\times\mathbf{E}+\mathbf{v}\times\frac{\partial \mathbf{E}}{\partial x}\right)\\ &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial \mathbf{E}}{\partial x}\end{aligned}$$

This means we only need to differentiate the electric field and later multiply by $\frac{1}{c^2}\mathbf{v}\times$ to get corresponding magnetic field values. Calcualting the derivatives of the electric potential come through the electric field as well. One has to remember that the electric field is a negative gradient of the electric potential but gradient is defined through the derivatives of the potential. And those derivatives are what we are looking for:

$$\begin{aligned}\mathbf{E} &= -\nabla\phi_E\\ &= -\frac{\partial\phi_E}{\partial x}\hat{\mathbf{i}}-\frac{\partial\phi_E}{\partial y}\hat{\mathbf{j}}-\frac{\partial\phi_E}{\partial z}\hat{\mathbf{k}}\\ \frac{\partial\phi_E}{\partial x} &= -E_x\end{aligned}$$

Now all that is left is taking derivatives of the electric field

$$\begin{aligned}\mathbf{r} &= (x-x_0)\hat{\mathbf{i}}+(y-y_0)\hat{\mathbf{j}}+(z-z_0)\hat{\mathbf{k}}\\ &= r_x\hat{\mathbf{i}}+r_y\hat{\mathbf{j}}+r_z\hat{\mathbf{k}}\\ \frac{\partial \mathbf{r}}{\partial x} &= \hat{\mathbf{i}}\\ \frac{\partial \mathbf{r}}{\partial y} &= \hat{\mathbf{j}}\\ \frac{\partial \mathbf{r}}{\partial z} &= \hat{\mathbf{k}}\\ r &= \sqrt{r_x^2+r_y^2+r_z^2}\\ \frac{\partial r}{\partial x} &= \frac{r_x}{r}\\ \frac{\partial r}{\partial y} &= \frac{r_y}{r}\\ \frac{\partial r}{\partial z} &= \frac{r_z}{r}\end{aligned}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$$

$$\frac{\partial \mathbf{E}}{\partial x} = \frac{q}{4\pi\varepsilon_0}\frac{\partial \frac{\mathbf{r}}{r^3}}{\partial x}$$

$$\frac{\partial \frac{\mathbf{r}}{r^3}}{\partial x} = \frac{r^3\hat{\mathbf{i}} - 3rr_x\mathbf{r}}{r^6}$$

$$= \frac{r^2\hat{\mathbf{i}} - 3r_x\mathbf{r}}{r^5}$$

$$\frac{\partial \frac{\mathbf{r}}{r^3}}{\partial y} = \frac{r^2\hat{\mathbf{j}} - 3r_y\mathbf{r}}{r^5}$$

$$\frac{\partial \frac{\mathbf{r}}{r^3}}{\partial z} = \frac{r^2\hat{\mathbf{k}} - 3r_z\mathbf{r}}{r^5}$$

$$\frac{\partial \mathbf{E}}{\partial x} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left(r^2\hat{\mathbf{i}} - 3r_x\mathbf{r}\right)$$

$$\frac{\partial \mathbf{E}}{\partial y} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left(r^2\hat{\mathbf{j}} - 3r_y\mathbf{r}\right)$$

$$\frac{\partial \mathbf{E}}{\partial z} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left(r^2\hat{\mathbf{k}} - 3r_z\mathbf{r}\right)$$

$$\frac{\partial \frac{1}{r^3}}{\partial y}\hat{\mathbf{i}} = -\frac{3r_y}{r^5}\hat{\mathbf{i}}$$

$$\frac{\partial \frac{1}{r^3}}{\partial z}\hat{\mathbf{i}} = -\frac{3r_z}{r^5}\hat{\mathbf{i}}$$

$$\frac{\partial \frac{1}{r^3}}{\partial z}\hat{\mathbf{j}} = -\frac{3r_z}{r^5}\hat{\mathbf{j}}$$

$$\frac{\partial \frac{3r_x}{r^5}}{\partial y} = \frac{-3r_x 5r^3 r_y}{r^{10}}$$

$$= -\frac{15r_x r_y}{r^7}$$

$$\frac{\partial \frac{3r_x}{r^5}}{\partial z} = -\frac{15r_x r_z}{r^7}$$

$$\frac{\partial \frac{3r_y}{r^5}}{\partial z} = -\frac{15r_y r_z}{r^7}$$

$$\frac{\partial \frac{3r_x}{r^5}\mathbf{r}}{\partial y} = -\frac{15r_x r_y}{r^7}\mathbf{r} + \frac{3r_x}{r^5}\hat{\mathbf{j}}$$

$$
\begin{aligned}
&= \frac{3r_x r^2\hat{\mathbf{j}} - 15 r_x r_y \mathbf{r}}{r^7} \\
\frac{\partial \frac{3r_x}{r^5}\mathbf{r}}{\partial z} &= \frac{3r_x r^2\hat{\mathbf{k}} - 15 r_x r_z \mathbf{r}}{r^7} \\
\frac{\partial \frac{3r_y}{r^5}\mathbf{r}}{\partial z} &= \frac{3r_y r^2\hat{\mathbf{k}} - 15 r_y r_z \mathbf{r}}{r^7}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial x \partial y} &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{3r_y}{r^5}\hat{\mathbf{i}} - \frac{3r_x r^2\hat{\mathbf{j}} - 15 r_x r_y \mathbf{r}}{r^7}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\frac{3}{r^7}\left(5r_x r_y \mathbf{r} - r^2 r_y \hat{\mathbf{i}} - r^2 r_x \hat{\mathbf{j}}\right) \\
\frac{\partial^2 \mathbf{E}}{\partial x \partial z} &= \frac{q}{4\pi\varepsilon_0}\frac{3}{r^7}\left(5r_x r_z \mathbf{r} - r^2 r_z \hat{\mathbf{i}} - r^2 r_x \hat{\mathbf{k}}\right) \\
\frac{\partial^2 \mathbf{E}}{\partial y \partial z} &= \frac{q}{4\pi\varepsilon_0}\frac{3}{r^7}\left(5r_y r_z \mathbf{r} - r^2 r_z \hat{\mathbf{j}} - r^2 r_y \hat{\mathbf{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \frac{r_y}{r^5}\hat{\mathbf{i}}}{\partial z} &= -\frac{5r_y r_z}{r^7}\hat{\mathbf{i}} \\
\frac{\partial \frac{r_x}{r^5}\hat{\mathbf{j}}}{\partial z} &= -\frac{5r_x r_z}{r^7}\hat{\mathbf{j}} \\
\frac{\partial \frac{5r_x r_y \mathbf{r}}{r^7}}{\partial z} &= \frac{5r_x r_y r^7\hat{\mathbf{k}} - 5r_x r_y 7 r^5 r_z \mathbf{r}}{r^{14}} \\
&= \frac{5r_x r_y r^2\hat{\mathbf{k}} - 35 r_x r_y r_z \mathbf{r}}{r^9}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \mathbf{E}}{\partial x \partial y \partial z} &= \frac{q}{4\pi\varepsilon_0}3\left(\frac{5r_y r_z}{r^7}\hat{\mathbf{i}} + \frac{5r_x r_z}{r^7}\hat{\mathbf{j}} + \frac{5r_x r_y}{r^7}\hat{\mathbf{k}} - \frac{35 r_x r_y r_z}{r^9}\mathbf{r}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\frac{15}{r^9}\left(r_y r_z r^2\hat{\mathbf{i}} + r_x r_z r^2\hat{\mathbf{j}} + r_x r_y r^2\hat{\mathbf{k}} - 7 r_x r_y r_z \mathbf{r}\right)
\end{aligned}
$$

3 Results

$$
\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3} \\
\frac{\partial \mathbf{E}}{\partial x} &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left(r^2\hat{\mathbf{i}} - 3r_x \mathbf{r}\right) \\
\frac{\partial \mathbf{E}}{\partial y} &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left(r^2\hat{\mathbf{j}} - 3r_y \mathbf{r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{E}}{\partial z} &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left(r^2\hat{\mathbf{k}} - 3r_z\mathbf{r}\right)\\
\frac{\partial^2 \mathbf{E}}{\partial x\partial y} &= \frac{q}{4\pi\varepsilon_0}\frac{3}{r^7}\left(5r_xr_y\mathbf{r} - r^2r_y\hat{\mathbf{i}} - r^2r_x\hat{\mathbf{j}}\right)\\
\frac{\partial^2 \mathbf{E}}{\partial x\partial z} &= \frac{q}{4\pi\varepsilon_0}\frac{3}{r^7}\left(5r_xr_z\mathbf{r} - r^2r_z\hat{\mathbf{i}} - r^2r_x\hat{\mathbf{k}}\right)\\
\frac{\partial^2 \mathbf{E}}{\partial y\partial z} &= \frac{q}{4\pi\varepsilon_0}\frac{3}{r^7}\left(5r_yr_z\mathbf{r} - r^2r_z\hat{\mathbf{j}} - r^2r_y\hat{\mathbf{k}}\right)\\
\frac{\partial^3 \mathbf{E}}{\partial x\partial y\partial z} &= \frac{q}{4\pi\varepsilon_0}\frac{15}{r^9}\left(r_yr_zr^2\hat{\mathbf{i}} + r_xr_zr^2\hat{\mathbf{j}} + r_xr_yr^2\hat{\mathbf{k}} - 7r_xr_yr_z\mathbf{r}\right)\\
\phi_E &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\\
\frac{\partial \phi_E}{\partial x} &= -\mathbf{E}\cdot\hat{\mathbf{i}}\\
\frac{\partial \phi_E}{\partial y} &= -\mathbf{E}\cdot\hat{\mathbf{j}}\\
\frac{\partial \phi_E}{\partial z} &= -\mathbf{E}\cdot\hat{\mathbf{k}}\\
\frac{\partial^2 \phi_E}{\partial x\partial y} &= -\frac{\partial \mathbf{E}}{\partial y}\cdot\hat{\mathbf{i}}\\
\frac{\partial^2 \phi_E}{\partial x\partial z} &= -\frac{\partial \mathbf{E}}{\partial z}\cdot\hat{\mathbf{i}}\\
\frac{\partial^2 \phi_E}{\partial y\partial z} &= -\frac{\partial \mathbf{E}}{\partial z}\cdot\hat{\mathbf{j}}\\
\frac{\partial^3 \phi_E}{\partial x\partial y\partial z} &= -\frac{\partial^2 \mathbf{E}}{\partial y\partial z}\cdot\hat{\mathbf{i}}\\
\mathbf{B} &= \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\\
\frac{\partial \mathbf{B}}{\partial x} &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial \mathbf{E}}{\partial x}\\
\frac{\partial \mathbf{B}}{\partial y} &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial \mathbf{E}}{\partial y}\\
\frac{\partial \mathbf{B}}{\partial z} &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial \mathbf{E}}{\partial z}\\
\frac{\partial^2 \mathbf{B}}{\partial x\partial y} &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial^2 \mathbf{E}}{\partial x\partial y}\\
\frac{\partial^2 \mathbf{B}}{\partial x\partial z} &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial^2 \mathbf{E}}{\partial x\partial z}\\
\frac{\partial^2 \mathbf{B}}{\partial y\partial z} &= \frac{1}{c^2}\mathbf{v}\times\frac{\partial^2 \mathbf{E}}{\partial y\partial z}
\end{aligned}
$$

$$\frac{\partial^3 \mathbf{B}}{\partial x \partial y \partial z} = \frac{1}{c^2}\mathbf{v} \times \frac{\partial^3 \mathbf{E}}{\partial x \partial y \partial z}$$

4 Disclaimer

The author of this document is not a physicist nor a mathematician and so is an amateur. The information in this document is provided in the hope it will be useful, but the author takes no responsibility for any errors or problems you may encounter.

References

[1] Introduction to Electrodynamics, David J. Griffiths, example 10.4