Electric potential and electric field within a sphere of charge

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July 03, 2008

1 Sphere of charge

Let the charge density within the sphere of radius R be following:

$$\rho(r) = k\left[1-\left(\frac{r}{R}\right)^2\right]^\alpha$$

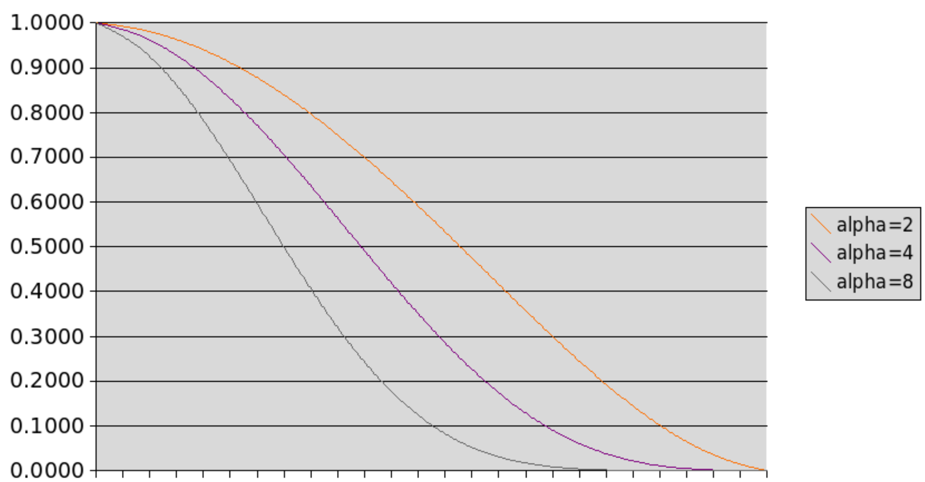

Figure 1: Charge density

Lets derive how the charge density factor k and charge at radius l relate in case the factor $\alpha = 2$:

$$q(l) = \int_0^l 4\pi r^2 \rho(r) dr$$

$$
\begin{aligned}
&= 4\pi k \int_0^l r^2 \left[1-\left(\frac{r}{R}\right)^2\right]^2 dr \\
&= 4\pi k \int_0^l \left(r^2 - \frac{2r^4}{R^2} + \frac{r^6}{R^4}\right) dr \\
&= 4\pi k \left(\frac{l^3}{3} - \frac{2r^5}{5R^2} + \frac{r^7}{7R^4}\right) \\
Q &= 4\pi k \left(\frac{R^3}{3} - \frac{2R^3}{5} + \frac{R^3}{7}\right) \\
&= 4\pi k \frac{8R^3}{105} \\
k &= \frac{Q}{4\pi}\frac{105}{8R^3} \\
q(r) &= Q\left(\frac{35r^3}{8R^3} - \frac{21r^5}{4R^5} + \frac{15r^7}{8R^7}\right)
\end{aligned}
$$

Now lets find the electric field using the Gauss's law:

$$
\begin{aligned}
\mathbf{\Phi} &= \oint_S \mathbf{E}\cdot d\mathbf{A} \\
&= \frac{q(r)}{\varepsilon_0} \\
&= E(r)4\pi r^2 \\
E(r) &= \frac{q(r)}{\varepsilon_0}\frac{1}{4\pi r^2} \\
&= \frac{Q}{4\pi\varepsilon_0}\left(\frac{35r}{8R^3} - \frac{21r^3}{4R^5} + \frac{15r^5}{8R^7}\right)
\end{aligned}
$$

Attempting to calculate the electric potential ϕ_E:

$$
\begin{aligned}
\phi_E(r) &= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} + \Delta\varphi(r) \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} - \int_R^r E(l)dl \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} - \frac{Q}{4\pi\varepsilon_0}\left(\frac{35l^2}{16R^3} - \frac{21l^4}{16R^5} + \frac{5l^6}{16R^7}\right)\Bigg|_R^r \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} - \frac{Q}{4\pi\varepsilon_0}\left(\frac{35r^2}{16R^3} - \frac{21r^4}{16R^5} + \frac{5r^6}{16R^7} - \frac{19}{16R}\right) \\
&= \frac{Q}{4\pi\varepsilon_0}\left(\frac{35}{16R} - \frac{35r^2}{16R^3} + \frac{21r^4}{16R^5} - \frac{5r^6}{16R^7}\right)
\end{aligned}
$$

2 Field and potential derivatives

So the e-field is:

$$\begin{aligned}\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{35r}{8R^3}-\frac{21r^3}{4R^5}+\frac{15r^5}{8R^7}\right)\hat{\mathbf{r}} \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{35}{8R^3}-\frac{21r^2}{4R^5}+\frac{15r^4}{8R^7}\right)\mathbf{r}\end{aligned}$$

Some helpers:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial\sqrt{(x-x_0)^2+(y-x_0)^2+(z-x_0)^2}}{\partial x} \\ &= \frac{x-x_0}{r} = \frac{r_x}{r} \\ M &= \frac{35}{8R^3}-\frac{21r^2}{4R^5}+\frac{15r^4}{8R^7} \\ \frac{\partial M}{\partial x} &= r_x\left(-\frac{21}{2R^5}+\frac{15r^2}{2R^7}\right) \\ &= r_xN \\ N &= -\frac{21}{2R^5}+\frac{15r^2}{2R^7} \\ \frac{\partial N}{\partial y} &= r_y\frac{15}{R^7} \\ &= r_yK \\ K &= \frac{15}{R^7} \\ \frac{\partial K}{\partial z} &= 0\end{aligned}$$

With helpers the e-field is:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}M\mathbf{r}$$

Lets take first order derivatives

$$\begin{aligned}\frac{\partial\mathbf{E}}{\partial x} &= \frac{Q}{4\pi\varepsilon_0}\left[r_xN\mathbf{r}+M\hat{\mathbf{i}}\right] \\ \frac{\partial\mathbf{E}}{\partial y} &= \frac{Q}{4\pi\varepsilon_0}\left[r_yN\mathbf{r}+M\hat{\mathbf{j}}\right] \\ \frac{\partial\mathbf{E}}{\partial z} &= \frac{Q}{4\pi\varepsilon_0}\left[r_zN\mathbf{r}+M\hat{\mathbf{k}}\right]\end{aligned}$$

Second order derivatives

$$
\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial x \partial y} &= \frac{Q}{4\pi\varepsilon_0}\left[r_x r_y K\mathbf{r} + r_y N\hat{\mathbf{i}} + r_x N\hat{\mathbf{j}}\right] \\
\frac{\partial^2 \mathbf{E}}{\partial x \partial z} &= \frac{Q}{4\pi\varepsilon_0}\left[r_x r_z K\mathbf{r} + r_z N\hat{\mathbf{i}} + r_x N\hat{\mathbf{k}}\right] \\
\frac{\partial^2 \mathbf{E}}{\partial y \partial z} &= \frac{Q}{4\pi\varepsilon_0}\left[r_y r_z K\mathbf{r} + r_z N\hat{\mathbf{j}} + r_y N\hat{\mathbf{k}}\right]
\end{aligned}
$$

Third order derivative

$$
\frac{\partial^3 \mathbf{E}}{\partial x \partial y \partial z} = \frac{Q}{4\pi\varepsilon_0}\left[r_y r_z K\hat{\mathbf{i}} + r_x r_z K\hat{\mathbf{j}} + r_x r_y K\hat{\mathbf{k}}\right]
$$

And finally lets try to take the first order derivatives for the potential

$$
\begin{aligned}
\frac{\partial \phi_E}{\partial x} &= -\frac{Q}{4\pi\varepsilon_0} r_x M \\
\frac{\partial \phi_E}{\partial y} &= -\frac{Q}{4\pi\varepsilon_0} r_y M \\
\frac{\partial \phi_E}{\partial z} &= -\frac{Q}{4\pi\varepsilon_0} r_z M
\end{aligned}
$$

3 Results

$$
\begin{aligned}
\rho(r) &= \frac{Q}{4\pi}\frac{105}{8R^3}\left[1 - \left(\frac{r}{R}\right)^2\right]^2 \\
M &= \frac{35}{8R^3} - \frac{21r^2}{4R^5} + \frac{15r^4}{8R^7} \\
N &= -\frac{21}{2R^5} + \frac{15r^2}{2R^7} \\
K &= \frac{15}{R^7} \\
\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0} M\mathbf{r} \\
\frac{\partial \mathbf{E}}{\partial x} &= \frac{Q}{4\pi\varepsilon_0}\left(r_x N\mathbf{r} + M\hat{\mathbf{i}}\right) \\
\frac{\partial \mathbf{E}}{\partial y} &= \frac{Q}{4\pi\varepsilon_0}\left(r_y N\mathbf{r} + M\hat{\mathbf{j}}\right) \\
\frac{\partial \mathbf{E}}{\partial z} &= \frac{Q}{4\pi\varepsilon_0}\left(r_z N\mathbf{r} + M\hat{\mathbf{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial x \partial y} &= \frac{Q}{4\pi\varepsilon_0}\left(r_x r_y K \mathbf{r} + r_y N \hat{\mathbf{i}} + r_x N \hat{\mathbf{j}}\right) \\
\frac{\partial^2 \mathbf{E}}{\partial x \partial z} &= \frac{Q}{4\pi\varepsilon_0}\left(r_x r_z K \mathbf{r} + r_z N \hat{\mathbf{i}} + r_x N \hat{\mathbf{k}}\right) \\
\frac{\partial^2 \mathbf{E}}{\partial y \partial z} &= \frac{Q}{4\pi\varepsilon_0}\left(r_y r_z K \mathbf{r} + r_z N \hat{\mathbf{j}} + r_y N \hat{\mathbf{k}}\right) \\
\frac{\partial^3 \mathbf{E}}{\partial x \partial y \partial z} &= \frac{Q}{4\pi\varepsilon_0}\left(r_y r_z K \hat{\mathbf{i}} + r_x r_z K \hat{\mathbf{j}} + r_x r_y K \hat{\mathbf{k}}\right) \\
\phi_E &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{35}{16R} - \frac{35r^2}{16R^3} + \frac{21r^4}{16R^5} - \frac{5r^6}{16R^7}\right) \\
\frac{\partial \phi_E}{\partial x} &= -\frac{Q}{4\pi\varepsilon_0} r_x M \\
\frac{\partial \phi_E}{\partial y} &= -\frac{Q}{4\pi\varepsilon_0} r_y M \\
\frac{\partial \phi_E}{\partial z} &= -\frac{Q}{4\pi\varepsilon_0} r_z M
\end{aligned}
$$

4 Disclaimer

The author of this document is not a physicist nor a mathematician and so is an amateur. The information in this document is provided in the hope it will be useful, but the author takes no responsibility for any errors or problems you may encounter.

5 History

The original document from January 11 contained a serious error in the equations.